Exercise 1. We trivially have

$$|\langle \delta_0', \varphi \rangle| = |-\varphi'(0)| = |\varphi'(0)|,$$

so the definition is satisfied with $C = 1$ and $m = 1$. We can also easily show that δ_0' is a distribution of order 1, and it is easy to check that $\mathrm{supp}(\delta_0') = \{0\}$.

Exercise 2. By definition, we have $T_n \underset{n\to\infty}{\longrightarrow} T \in \mathscr{D}'(\Omega)$ if and only if for all $\varphi \in \mathscr{D}(\Omega)$, we have

$$T_n(\varphi) \underset{n\to\infty}{\longrightarrow} T(\varphi).$$

In particular, if $\psi = D^\alpha \varphi$, then $\psi \in \mathscr{D}(\Omega)$, which implies that

$$D^\alpha T_n(\varphi) = (-1)^{|\alpha|} T_n(D^\alpha \varphi) = (-1)^{|\alpha|} T_n(\psi) \underset{n\to\infty}{\longrightarrow} (-1)^{|\alpha|} T(\psi) = (-1)^{|\alpha|} T(D^\alpha \varphi) = D^\alpha T(\varphi)$$

by definition de $D^\alpha T$.

Exercise 3. For all $\varphi \in \mathscr{D}(\mathbb{R})$, we have

$$T_n(\varphi) = \delta_{\frac{1}{n}}(\varphi) = \varphi\left(\frac{1}{n}\right) \underset{n\to\infty}{\longrightarrow} \varphi(0) = \delta_0(\varphi)$$

as $\varphi \in C^0(\mathbb{R})$. Therefore, we have

$$T_n \underset{n\to\infty}{\longrightarrow} \delta_0 \qquad \text{dans } \mathscr{D}'(\mathbb{R}).$$

We have

$$\varphi(x) = \varphi(0) + \varphi'(0)x + O(x^2). \tag{1}$$

Therefore, we get

$$\frac{\varphi(x) - \varphi\left(\frac{x}{2}\right)}{x} = \frac{1}{2}\varphi'(0) + O(|x|),$$

which shows that

$$\begin{aligned} S_n(\varphi) = n(T_n - T_{2n})(\varphi) &= n\left(\varphi\left(\frac{1}{n}\right) - \varphi\left(\frac{1}{2n}\right)\right) = \frac{\varphi\left(\frac{1}{n}\right) - \varphi\left(\frac{1}{2n}\right)}{\frac{1}{n}} \\ &= \frac{1}{2}\varphi'(0) + O\left(\frac{1}{n}\right) \underset{n\to\infty}{\longrightarrow} \frac{1}{2}\varphi'(0) = -\frac{1}{2}\delta_0'(\varphi), \end{aligned}$$

and finally, we have

$$S_n \underset{n\to\infty}{\longrightarrow} -\frac{1}{2}\delta_0' \qquad \text{dans } \mathscr{D}'(\mathbb{R}).$$

Exercise 4. 1. For all $\varphi \in \mathscr{D}(\mathbb{R})$, we have

$$\begin{aligned}\langle e^x\cdot\delta_0'',\varphi\rangle &= \langle\delta_0'',e^x\varphi(x)\rangle = \frac{d^2}{dx^2}\left(e^x\varphi(x)\right)_{|x=0} = \varphi(0)+2\,\varphi'(0)+\varphi''(0)\\ &= (\delta_0-2\,\delta_0'+\delta_0'')(\varphi),\end{aligned}$$

which shows that

$$e^x\cdot\delta_0'' = \delta_0-2\,\delta_0'+\delta_0''.$$

2. We have

$$\begin{aligned}\langle f_{a,b}',\varphi\rangle &= -\langle f_{a,b},\varphi'\rangle = -\int_{-\infty}^{0}\log(-b\,x)\varphi'(x)dx-\int_0^{\infty}\log(a\,x)\varphi'(x)dx\\ &= -\int_{\mathbb{R}}\log|x|\varphi'(x)dx-\int_{-\infty}^{0}\log(b)\varphi'(x)dx-\int_0^{\infty}\log(a)\varphi'(x)dx\\ &= -\int_{\mathbb{R}}\log|x|\varphi'(x)dx-\log(b)\left[\varphi(x)\right]_{-\infty}^{0}-\log(a)\left[\varphi(x)\right]_0^{\infty}\\ &= -\int_{\mathbb{R}}\log|x|\varphi'(x)+\log\left(\frac{a}{b}\right)\varphi(0).\end{aligned}\tag{2}$$

For all $\varepsilon>0$, we have

$$\begin{aligned}\int_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]}\log|x|\varphi'(x)dx &= -\int_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]}\frac{\varphi(x)}{x}dx+\Big[\log|x|\varphi(x)\Big]_{-\infty}^{-\varepsilon}+\Big[\log|x|\varphi(x)\Big]_{\varepsilon}^{\infty}\\ &= -\int_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]}\frac{\varphi(x)}{x}dx+\log(\varepsilon)(\varphi(-\varepsilon)-\varphi(\varepsilon)).\end{aligned}$$

As

$$\lim_{\varepsilon\to0}\varepsilon\log(\varepsilon)=0,$$

and using (1)

$$\varphi(-\varepsilon)-\varphi(\varepsilon) = -2\varphi'(0)\varepsilon+O(\varepsilon^2),$$

we get

$$\lim_{\varepsilon\to0}\log(\varepsilon)(\varphi(-\varepsilon)-\varphi(\varepsilon))=0,$$

which shows that

$$\begin{aligned}\int_{\mathbb{R}}\log|x|\varphi'(x)dx &= \lim_{\varepsilon\to0}\int_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]}\log|x|\varphi'(x)dx\\ &= \lim_{\varepsilon\to0}\left(-\int_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]}\frac{\varphi(x)}{x}dx+\log(\varepsilon)(\varphi(-\varepsilon)-\varphi(\varepsilon))\right)\\ &= -\left\langle \mathrm{v.p.}\frac{1}{x},\varphi\right\rangle.\end{aligned}\tag{3}$$

Finally, using (2) and (3), we deduce that for all $\varphi\in\mathscr{D}(\mathbb{R})$, we have

$$\langle f_{a,b}',\varphi\rangle = \left\langle \mathrm{v.p.}\frac{1}{x},\varphi\right\rangle+\log\left(\frac{a}{b}\right)\varphi(0).$$

In other words, we have established that

$$f_{a,b}' = \mathrm{v.p.}\frac{1}{x}+\log\left(\frac{a}{b}\right)\delta_0.$$

Exercise 5. Let $\varphi \in \mathscr{D}(\mathbb{R})$ and $R > 0$ be such that $\varphi(x) = 0$ for all $|x| > R$. Then, we have

$$\begin{aligned}
T_n(\varphi) &= \int_{-R}^{\varepsilon_n^-} \frac{\varphi(x)}{x} dx + \int_{\varepsilon_n^+}^{R} \frac{\varphi(x)}{x} dx \\
&= \int_{-R}^{-\varepsilon_n^-} \frac{\varphi(x) - \varphi(0)}{x} dx + \int_{\varepsilon_n^+}^{R} \frac{\varphi(x) - \varphi(0)}{x} dx + \int_{-R}^{-\varepsilon_n^-} \frac{\varphi(0)}{x} dx + \int_{\varepsilon_n^+}^{R} \frac{\varphi(0)}{x} dx \\
&= \int_{-R}^{-\varepsilon_n^-} \frac{\varphi(x) - \varphi(0)}{x} dx + \int_{\varepsilon_n^+}^{R} \frac{\varphi(x) - \varphi(0)}{dx} - \varphi(0) \log\left(\frac{R}{\varepsilon_n^-}\right) + \varphi(0) \log\left(\frac{R}{\varepsilon_n^+}\right) \\
&= \int_{-R}^{-\varepsilon_n^-} \frac{\varphi(x) - \varphi(0)}{x} dx + \int_{\varepsilon_n^+}^{R} \frac{\varphi(x) - \varphi(0)}{dx} - \log\left(\frac{\varepsilon_n^+}{\varepsilon_n^-}\right) \varphi(0) \\
&\underset{n\to\infty}{\longrightarrow} \int_{-R}^{R} \frac{\varphi(x) - \varphi(0)}{x} dx - \log(a)\varphi(0) \\
&= \left\langle \text{v.p.}\frac{1}{x} - \log(a)\delta_0, \varphi \right\rangle,
\end{aligned}$$

which shows that

$$\lim_{n\to\infty} T_n = \text{v.p.}\frac{1}{x} + \log(a)\delta_0 \qquad \text{dans } \mathscr{D}'(\mathbb{R}).$$

Exercise 6. Using the formula of jumps, as g is continuous in 0 and C^1 in $\mathbb{R}^*$, we deduce that $[g]' = [g'] = [H]$, where H is the Heaviside function. And we have already showed in class that $H' = \delta_0$, which completes the proof. We can also compute directly :

$$\langle Lg, \varphi \rangle = \langle g, \varphi'' \rangle = \int_0^\infty x\, \varphi''(x) dx = -\int_0^\infty \varphi'(x) dx = \varphi(0).$$

Exercise 7. We first check that $\Delta \log = 0$ in $\mathbb{R}^2 \setminus \{0\}$. Using polar coordinates, this is easy since

$$\Delta = \partial_r^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\partial_\theta^2.$$

Therefore we have

$$\begin{aligned}
&\partial_r \log|x| = \partial_r \log(r) = \frac{1}{r} \\
&\partial_r^2 \log|x| = -\frac{1}{r^2},
\end{aligned}$$

which shows that

$$\Delta \log|x| = \left(\partial_r^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\partial_\theta^2\right) \log r = -\frac{1}{r^2} + \frac{1}{r}\left(\frac{1}{r}\right) + 0 = 0.$$

Integrating by parts, we get

$$\begin{aligned}
&\int_{\mathbb{R}^2 \setminus \overline{B}_\varepsilon(0)} \log|x| \Delta\varphi(x) dx = \int_{\mathbb{R}^2} \varphi(x) \Delta \log|x|\, dx + \int_{\partial(\mathbb{R}^2 \setminus \overline{B}_\varepsilon(0))} \left(\log|x|\, \partial_\nu \varphi - \partial_\nu(\log|x|)\varphi\right) dl \\
&= \int_{\partial B(0,\varepsilon)} \left((\partial_r \log r)\varphi - \log r(\partial_\nu \varphi)\right) dl.
\end{aligned}$$

Beware of the sign change! The function φ is smooth, and since it is bounded in 0, we also have

$$\left|\int_{\partial B(0,\varepsilon)} \log r\,\partial_\nu \varphi\,dl\right| \leq \|\nabla\varphi\|_{\mathrm{L}^\infty(\mathbb{R}^2)} |\log(\varepsilon)| \int_{\partial B(0,\varepsilon)} dl = 2\pi\varepsilon\,|\log(\varepsilon)|\,\|\nabla\varphi\|_{\mathrm{L}^\infty(\mathbb{R}^2)} \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

On the other hand, as $x \to 0$, we have

$$\varphi(x) = \varphi(0) + O(\varepsilon),$$

which shows that

$$\int_{\partial B(0,\varepsilon)} (\partial_r \log r)\varphi\,dl = \int_{\partial B(0,\varepsilon)} \frac{\varphi(x)}{\varepsilon} dl = 2\pi\varphi(0) + O(\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} 2\pi\varphi(0).$$

Finally, we get

$$\langle \Delta G, \varphi\rangle = \langle G, \Delta\varphi\rangle = \frac{1}{2\pi}\int_{\mathbb{R}^2} \log|x|\Delta\varphi(x)dx = \frac{1}{2\pi}\lim_{\varepsilon\to 0}\int_{\mathbb{R}^2\setminus\overline{B}_\varepsilon(0)} \log|x|\varphi(x) = \varphi(0),$$

which shows by definition that

$$\Delta G = \delta_0 \qquad \text{dans } \mathscr{D}'(\mathbb{R}^2).$$